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HYPERGEOMETRIC FUNCTIONS AND ASSOCIATED FAMILIES OF MEROMORPHICALLY STARLIKE AND CONVEX FUNCTIONS

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Abstract. For hypergeometric function ${}_2F_1(r,s;t;z)$, conditions on r, s and t are investigate so that ${}_2F_1(r,s;t;z)/z$ is meromorphically starlike and convex functions in the punctured disk. Further, integral operators related to the hypergeometric function are also examined.

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1. Introduction

Let Σ denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic in the punctured open unit disk $\mathbb{U}_0=\{z:0<|z|<1\}$. A function $f\in\Sigma$ is said to be meromorphically starlike of order α, if it satisfies

$$-\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \quad (0\le\alpha<1).$$

A function $f\in\Sigma$ is said to be meromorphically convex of order α, if it satisfies

$$-\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\alpha \quad (0\le\alpha<1).$$

We denote by $\Sigma^*(\alpha)$ and $\Sigma_k(\alpha)$ the subclasses of Σ consisting of all meromorphically starlike and convex functions of order α, respectively.

Let Σ_p denote the subclass of Σ consisting of functions of the form

$$f(z)=\frac{1}{z}+\sum_{n=0}^{\infty}a_nz^n \ (a_n\ge 0) \tag{1.2}$$

and let $\Sigma_p^*(\alpha)=\Sigma^*(\alpha)\cap\Sigma_p$. Also let Σ_T denote the subclass of Σ consisting of functions of the form

$$f(z)=\frac{1}{z}-\sum_{n=0}^{\infty}a_nz^n \ (a_n\ge 0) \tag{1.3}$$

and let $\Sigma_{k,T}(\alpha)=\Sigma_k(\alpha)\cap\Sigma_T$. For various other interesting development involving functions in the classes $\Sigma_p^*(\alpha)$ and $\Sigma_{k,T}(\alpha)$, the reader may be referred (for example) to the works of Mogra et.al [5] and Uralegaddi and Ganigi [9].

The hypergeometric function ${}_2F_1(r,s;t;z)$ is given as a power series, converging in the open unit disk $\mathbb{U} = \mathbb{U}_0 \cup \{0\}$, in the following way:

$$ {}_2F_1(z) \equiv {}_2F_1(r,s;t;z) := \sum_{n=0}^{\infty} \frac{(r)_n(s)_n}{(t)_n(1)_n} z^n, $$

where r, s and t are complex numbers with $t \neq 0, -1, -2, \cdots$, and $(\lambda)_n$ denotes the Pochhammer symbol (or the shifted factorial) defined, in terms of the Gamma function Γ, by

$$ \begin{aligned} (\lambda)_n &:= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \\ &= \begin{cases} 1 & \text{if } n=0 \text{ and } \lambda \in \mathbb{C}\backslash\{0\} \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & \text{if } n \in \mathbb{N} = \{1,2,\cdots\} \text{ and } \lambda \in \mathbb{C}. \end{cases} \end{aligned} $$

We note that ${}_2F_1(r,s;t;1)$ converges for $\mathrm{Re}(t-r-s) > 0$ and is represented by

$$ {}_2F_1(r,s;t;1) = \frac{\Gamma(t)\Gamma(t-r-s)}{\Gamma(t-r)\Gamma(t-s)}. \tag{1.4} $$

By using the hypergeometric function ${}_2F_1(z)$, we now introduce two meromorphic functions as follows:

$$ L(z) \equiv L(r,s;t;z) := \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n+1}} z^n \quad (z \in \mathbb{U}_0). \tag{1.5} $$

and

$$ I(z) \equiv I(r,s;t;z) := \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n+2}} z^n \quad (z \in \mathbb{U}_0). \tag{1.6} $$

We note that

$$ L(z) = \frac{{}_2F_1(z)}{z} \text{ and } I(z) = \frac{\int_0^z {}_2F_1(t)dt}{z^2}. $$

There are several results for ${}_2F_1(z)$ in connection with various classes of functions (see [1, 2, 4, 6, 7]). Particularly, Silverman [8] determined necessary and sufficient conditions for $z{}_2F_1(z)$ to be in various subclasses of starlike and convex functions. Also, Liu and Srivastava [3] investigated some geometric properties for two novel families of meromorphically multivalent functions involving a linear operator $L(r,1;t;z)$.

In this paper, we will determine necessary and sufficient conditions for $L(z)$ and $I(z)$ to be in the classes $\Sigma_p^*(\alpha)$ and $\Sigma_{k,T}(\alpha)$ with appropriate restrictions on r, s and t.

2. MAIN RESULTS

Lemma 2.1. *A sufficient condition for the function f given by* (1.1) *to be in $\Sigma^*(\alpha)$ is that*

$$ \sum_{n=0}^{\infty} (n+\alpha)|a_n| \leq 1-\alpha, \quad (1/2 \leq \alpha < 1). \tag{2.1} $$

Further, a necessary and sufficient condition for the function f given by (1.2) *to be in $\Sigma_p^*(\alpha)$ is that the condition* (2.1) *is satisfied.*

Proof. Suppose that the inequality $\sum_{n=0}^{\infty}(n+\alpha)|a_n| \leq 1-\alpha$ holds. Then it suffices to show that

$$\left|\frac{zf'(z)}{f(z)}+1\right| < \left|\frac{zf'(z)}{f(z)}+2\alpha-1\right|. \tag{2.2}$$

For $0<|z|=r<1$, (2.2) is equivalent to

$$\begin{aligned} H(f,f') &= |zf'(z)+f(z)| - |zf'(z)+(2\alpha-1)f(z)| \\ &= \left|\sum_{n=0}^{\infty}(n+1)a_nz^n\right| - \left|2(1-\alpha)\frac{1}{z} - \sum_{n=0}^{\infty}(n+2\alpha-1)a_nz^n\right|, \end{aligned}$$

or

$$\begin{aligned} rH(f,f') &= \sum_{n=0}^{\infty}(n+1)|a_n|r^{n+1} - 2(1-\alpha) + \sum_{n=0}^{\infty}(n+2\alpha-1)|a_n|r^{n+1} \\ &= \sum_{n=0}^{\infty}2(n+\alpha)|a_n|r^{n+1} - 2(1-\alpha). \end{aligned}$$

Since the above inequality holds for all r $(0<r<1)$, letting $r\to 1^-$, we have

$$H(f,f') \leq \sum_{n=0}^{\infty}2(n+\alpha)|a_n| - 2(1-\alpha) \leq 0,$$

by (2.1). This completes the proof of the first part of Lemma 2.1. In order to prove the second part of Lemma 2.1, we assume that $f\in\Sigma_p^*(\alpha)$. Then, from (2.2), we have

$$\left|\frac{zf'(z)+f(z)}{zf'(z)+(2\alpha-1)f(z)}\right| = \left|\frac{\sum_{n=0}^{\infty}(n+1)a_nz^n}{2(\alpha-1)\frac{1}{z}+\sum_{n=0}^{\infty}(n+2\alpha-1)a_nz^n}\right| < 1.$$

Since $\mathrm{Re}(z)\leq|z|$ for all z,

$$\mathrm{Re}\left(\frac{\sum_{n=0}^{\infty}(n+1)a_nz^n}{2(1-\alpha)\frac{1}{z}-\sum_{n=0}^{\infty}(n+2\alpha-1)a_nz^n}\right) < 1. \tag{2.3}$$

Choose the values of z on the real axis so that $zf'(z)/f(z)$ is real. Upon clearing the denominator in (2.3) and letting $z\to 1^-$ through positive values, we obtain the condition (2.1). □

Lemma 2.2. *A sufficient condition for the function f is given by* (1.1) *to be in $\Sigma_k(\alpha)$ is that*

$$\sum_{n=0}^{\infty}n(n+\alpha)|a_n| \leq 1-\alpha \quad (0\leq\alpha<1). \tag{2.4}$$

Further, a necessary and sufficient condition for the function f is given by (2.4) *to be in $\Sigma_{k,T}(\alpha)$ is that the condition* (1.3) *is satisfied.*

Proof. Since the proof is similar to that of Lemma 2.1, we shall omit the detailed proof. □

Theorem 2.1. *If $r,s>0$, then a necessary and sufficient condition for $L(z)$, given by* (1.5), *to be in $\Sigma_p^*(\alpha)$ is that $t\geq rs/(1-\alpha)+r+s+1$.*

Proof. According to Lemma 2.1, we need to show that

$$\sum_{n=0}^{\infty}(n+\alpha)\frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n+1}} \leq 1-\alpha. \tag{2.5}$$

Now

$$\sum_{n=0}^{\infty}(n+\alpha)\frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n+1}} = \sum_{n=0}^{\infty}\frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n}} + (\alpha-1)\sum_{n=1}^{\infty}\frac{(r)_{n}(s)_{n}}{(t)_{n}(1)_{n}}. \tag{2.6}$$

Noting that $(\lambda)_{n+1} = \lambda(\lambda+1)_n$ and then applying (1.4), we may express (2.6) as

$$\begin{aligned}
&\frac{rs}{t}\sum_{n=0}^{\infty}\frac{(r+1)_n(s+1)_n}{(t+1)_n(1)_n} + (\alpha-1)\left(\sum_{n=0}^{\infty}\frac{(r)_n(s)_n}{(t)_n(1)_n} - 1\right)\\
&= \left(\frac{rs}{t-r-s-1} + \alpha - 1\right)\frac{\Gamma(t)\Gamma(t-r-s)}{\Gamma(t-r)\Gamma(t-s)} + 1 - \alpha.
\end{aligned}$$

Hence, (2.5) is equivalent to

$$\left(\frac{rs}{t-r-s-1} + \alpha - 1\right)\frac{\Gamma(t)\Gamma(t-r-s)}{\Gamma(t-r)\Gamma(t-s)} + 1 - \alpha \leq 1-\alpha. \tag{2.7}$$

Thus, (2.7) is valid if and only if $t \geq rs/(1-\alpha) + r + s + 1$. □

Theorem 2.2. *If $r, s > -1$, $t \geq r+s+2$ and $rs < 0$. A necessary and sufficient condition for $L(z)$, given by (1.5), to be in $\Sigma_{k,T}(\alpha)$ is that*

$$\frac{(r+1)(s+1)}{(t-r-s-2)_2} - (1-\alpha)\left(\frac{1}{t-r-s-1} - \frac{1}{rs}\right) \leq 0. \tag{2.8}$$

Proof. Since

$$L(z) = \frac{1}{z} - \left|\frac{rs}{t}\right|\sum_{n=0}^{\infty}\frac{(r+1)_n(s+1)_n}{(t+1)_n(1)_{n+1}}z^n,$$

according to Lemma 2.2, we must show that

$$\sum_{n=0}^{\infty}n(n+\alpha)\frac{(r+1)_n(s+1)_n}{(t+1)_n(1)_{n+1}} \leq \left|\frac{t}{rs}\right|(1-\alpha).$$

Writing $n(n+\alpha) = (n+1)^2 - (2-\alpha)(n+1) + (1-\alpha)$, we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty}n(n+\alpha)\frac{(r+1)_n(s+1)_n}{(t+1)_n(1)_{n+1}}\\
&= \frac{(r+1)(s+1)}{t+1}\sum_{n=0}^{\infty}\frac{(r+2)_n(s+2)_n}{(t+2)_n(1)_n} - (1-\alpha)\sum_{n=0}^{\infty}\frac{(r+1)_n(s+1)_n}{(t+1)_n(1)_n}\\
&\quad + \frac{(1-\alpha)t}{rs}\sum_{n=1}^{\infty}\frac{(r)_n(s)_n}{(t)_n(1)_n}\\
&= \frac{\Gamma(t+1)\Gamma(t-r-s)}{\Gamma(t-r)\Gamma(t-s)}\left(\frac{(r+1)(s+1)}{(t-r-s-1)(t-r-s-2)}\right.\\
&\quad \left. -(1-\alpha)\left(\frac{1}{t-r-s-1} - \frac{1}{rs}\right)\right) - \frac{(1-\alpha)t}{rs}.
\end{aligned}$$

The last expression is bounded above by $|t/(rs)|(1-\alpha)$ if and only if (2.8) holds. □

Theorem 2.3. *If $r, s > 0$ and $t > r+s+2$, then a necessary and sufficient condition for $G(z) = \frac{1}{z}(2 - {}_2F_1(z))$ to be in $\Sigma_{k,T}(\alpha)$ is that*

$$\frac{\Gamma(t)\Gamma(t-r-s)}{\Gamma(t-r)\Gamma(t-s)}\left(\frac{(r)_2(s)_2}{(1-\alpha)(t-r-s-2)_2} - \frac{rs}{t-r-s-1} + 1\right) \leq 2. \tag{2.9}$$

Proof. Since

$$G(z) = \frac{1}{z} - \sum_{n=0}^{\infty} \frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n+1}} z^n,$$

from Lemma 2.2, it suffices to show that

$$\sum_{n=0}^{\infty} n(n+\alpha)\frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n+1}} \leq 1-\alpha.$$

Now, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} n(n+\alpha)\frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n+1}} \\
&= \sum_{n=0}^{\infty} n\frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n}} - (1-\alpha)\sum_{n=0}^{\infty} \frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n}} + (1-\alpha)\sum_{n=0}^{\infty} \frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n+1}} \\
&= \frac{r(r+1)s(s+1)}{t(t+1)}\sum_{n=0}^{\infty} \frac{(r+2)_n(s+2)_n}{(t+2)_n(1)_n} - \frac{(1-\alpha)rs}{t}\sum_{n=0}^{\infty} \frac{(r+1)_n(s+1)_n}{(t+1)_n(1)_n} \\
&\quad +(1-\alpha)\left(\sum_{n=0}^{\infty} \frac{(r)_n(s)_n}{(t)_n(1)_n} - 1\right) \\
&= \frac{\Gamma(t)\Gamma(t-r-s)}{\Gamma(t-r)\Gamma(t-s)}\left(\frac{(r)_2(s)_2}{(t-r-s-2)_2} - \frac{(1-\alpha)rs}{t-r-s-1} + 1 - \alpha\right) - (1-\alpha).
\end{aligned}$$

This last expression is bounded above by $1-\alpha$ if and only if (2.9) holds. □

Theorem 2.4. *If $r, s > 1$ and $t > r+s$, then a necessary and sufficient condition for $I(z)$, given by (1.6), to be in $\Sigma_p^*(\alpha)$ is that*

$$\frac{\Gamma(t)\Gamma(t-r-s)}{\Gamma(t-r)\Gamma(t-s)}\left(\frac{(r-1)(s-1)}{(\alpha-2)(t-1)} + \frac{t-r-s}{t-1}\right) \geq 1 \tag{2.10}$$

Proof. In view of Lemma 2.1, we need only to show that

$$\sum_{n=0}^{\infty}(n+\alpha)\frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n+2}} \leq 1-\alpha.$$

Now

$$\begin{aligned}
&\sum_{n=0}^{\infty}(n+\alpha)\frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n+2}} \\
&= \sum_{n=0}^{\infty} \frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n+1}} + (\alpha-2)\sum_{n=0}^{\infty} \frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n+2}} \\
&= \frac{\Gamma(t)\Gamma(t-r-s)}{\Gamma(t-r)\Gamma(t-s)}\left(1 + \frac{(\alpha-2)(t-r-s)}{(r-1)(s-1)}\right) - \frac{(\alpha-2)(t-1)}{(r-1)(s-1)} + 1 - \alpha.
\end{aligned}$$

But this last expression is bounded above by $1-\alpha$ if and only if (2.10) holds. □

Theorem 2.5. *If $r,s>-1$ with $rs<0$ ($r\neq 1$ or $s\neq 1$) and $t>r+s+1$, then a necessary and sufficient condition for $I(z)$, given by (1.6), to be in $\Sigma_{k,T}(\alpha)$ is that*

(2.11)
$$\frac{\Gamma(t+1)\Gamma(t-r-s)}{\Gamma(t-r)\Gamma(t-s)}\left(\frac{1}{t-r-s-1}-\frac{(3-\alpha)}{rs}+\frac{2(2-\alpha)(t-r-s)}{(r-1)_2(s-1)_2}\right)\leq\frac{2(2-\alpha)(t-1)_2}{(r-1)_2(s-1)_2}.$$

Proof. Since
$$I(z)=\frac{1}{z}-\left|\frac{rs}{t}\right|\sum_{n=0}^{\infty}\frac{(r+1)_n(s+1)_n}{(t+1)_n(1)_{n+2}}z^n,$$
by Lemma 2.2, it is sufficient to show that
$$\sum_{n=0}^{\infty}n(n+\alpha)\frac{(r+1)_n(s+1)_n}{(t+1)_n(1)_{n+2}}\leq\left|\frac{t}{rs}\right|(1-\alpha).$$
Now
$$\begin{aligned}&\sum_{n=0}^{\infty}n(n+\alpha)\frac{(r+1)_n(s+1)_n}{(t+1)_n(1)_{n+2}}\\&=\sum_{n=0}^{\infty}(n+\alpha)\frac{(r+1)_n(s+1)_n}{(t+1)_n(1)_{n+1}}-2\sum_{n=0}^{\infty}(n+\alpha)\frac{(r+1)_n(s+1)_n}{(t+1)_n(1)_{n+2}}\\&=\sum_{n=0}^{\infty}\frac{(r+1)_n(s+1)_n}{(t+1)_n(1)_n}-(3-\alpha)\frac{t}{rs}\sum_{n=0}^{\infty}\frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n+1}}\\&\quad+\frac{2(2-\alpha)(t-1)_2}{(r-1)_2(s-1)_2}\sum_{n=0}^{\infty}\frac{(r-1)_{n+2}(s-1)_{n+2}}{(t-1)_{n+2}(1)_{n+2}}\\&=\frac{\Gamma(t+1)\Gamma(t-r-s)}{\Gamma(t-r)\Gamma(t-s)}\left(\frac{1}{t-r-s-1}-\frac{(3-\alpha)}{rs}+\frac{2(2-\alpha)(t-r-s)}{(r-1)_2(s-1)_2}\right)\\&\quad-\frac{2(2-\alpha)(t-1)_2}{(r-1)_2(s-1)_2}-\frac{(1-\alpha)t}{rs}.\end{aligned}$$
But this last expression is bounded above by $|t/(rs)|(1-\alpha)$ if and only if (2.11) holds. □

Theorem 2.6. *If $r,s>1$ and $t>r+s+1$, then a necessary and sufficient condition for $H(z)=\int_0^z(2-{}_2F_1(z))dz/z^2$ to be in $\Sigma_{k,T}(\alpha)$ is that*
$$\begin{aligned}&\frac{\Gamma(t)\Gamma(t-r-s)}{\Gamma(t-r)\Gamma(t-s)}\left(\frac{rs}{t-r-s-1}+\frac{2(2-\alpha)(t-r-s)}{(r-1)(s-1)}-3+\alpha\right)\\&\leq\frac{2(2-\alpha)(t-1)}{(r-1)(s-1)}+2(1-\alpha).\end{aligned}\tag{2.12}$$

Proof. The function H can be written by as
$$H(z)=\frac{1}{z}-\sum_{n=0}^{\infty}\frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n+2}}z^n.$$

According to Lemma 2.2, we will show that

$$\sum_{n=0}^{\infty} n(n+\alpha)\frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n+2}} \leq 1-\alpha.$$

Now

$$\begin{aligned}
&\sum_{n=0}^{\infty} n(n+\alpha)\frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n+2}} \\
&\sum_{n=0}^{\infty} (n+\alpha)\frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n+1}} - \sum_{n=0}^{\infty} 2(n+\alpha)\frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n+2}} \\
= \ &\sum_{n=0}^{\infty} \frac{rs}{t}\frac{(r+1)_n(s+1)_n}{(t+1)_n(1)_n} + (3-\alpha)\sum_{n=0}^{\infty} \frac{(r)_{n+1}(s)_{n+1}}{(t)_{n+1}(1)_{n+1}} \\
&+2(2-\alpha)\sum_{n=0}^{\infty} \frac{t-1}{(r-1)(s-1)}\frac{(r-1)_{n+2}(s-1)_{n+2}}{(t-1)_{n+2}(1)_{n+2}} \\
= \ &\frac{\Gamma(t)\Gamma(t-r-s)}{\Gamma(t-r)\Gamma(t-s)}\left(\frac{st}{t-r-s-1} - \frac{2(\alpha-2)(t-r-s)}{(r-1)(s-1)} + \alpha - 3\right) \\
&+\frac{2(\alpha-2)(t-1)}{(r-1)(s-1)} - 1 + \alpha.
\end{aligned}$$

But this last expression is bounded above by $1-\alpha$ if and only if (2.12) holds. □

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